Main Video on YouTube.

1 Introduction

We know, by Wolfram's and Alpha's powers that:

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \Rightarrow \sin\frac{2\pi}{5} = \frac{\sqrt{2}}{4}\sqrt{5+\sqrt{5}} \tag{1}$$

$$\cos 30^\circ = 2\cos^2 15^\circ - 1 \tag{2}$$

$$a_1 = 2x_1^2 - 1 \Rightarrow x_1 = \sqrt{\frac{a_1+1}{2}} \tag{3}$$

$$\cos 72^\circ = 2\cos^2 36^\circ - 1 \tag{4}$$

$$a_2 = 2x_2^2 - 1 \Rightarrow x_2 = \sqrt{\frac{a_2+1}{2}} = \sqrt{\frac{\frac{\sqrt{5}-1}{4}+1}{2}} \tag{5}$$

$$\cos 36^\circ = 2\cos^2 18^\circ - 1 \tag{6}$$

$$a_3 = 2x_3^2 - 1 \Rightarrow x_3 = \sqrt{\frac{a_3+1}{2}} \tag{7}$$

$$\cos 18^\circ = 2\cos^2 9^\circ - 1 \tag{8}$$

$$a_4 = 2x_4^2 - 1 \Rightarrow x_4 = \sqrt{\frac{a_4+1}{2}} \tag{9}$$

$$\cos 27^\circ = 4x_4^3 - 3x_4 = x_5 \tag{10}$$

$$\cos 30^\circ = 1 - 2\sin^2 15^\circ \tag{11}$$

$$a_1 = 1 - 2y_1^2 \Rightarrow y_1 = \sqrt{\frac{1-a_1}{2}};\ y_2 = \sqrt{\frac{1-a_2}{2}};\ y_3 = \sqrt{\frac{1-a_3}{2}};\ y_4 = \sqrt{\frac{1-a_4}{2}} \tag{12}$$

$$\sin 27^\circ = \sin 9^\circ(3 - 4\sin^2 9^\circ) \tag{13}$$

$$y_5 = y_4(3 - 4y_4^2) \tag{14}$$

$$\cos(30^\circ - 27^\circ) = \frac{\sqrt{3}}{2}\cdot x_5 + \frac{1}{2}\cdot y_5 = x_6 \tag{15}$$

$$\sin(30^\circ - 27^\circ) = \frac{1}{2}\cdot x_5 - \frac{\sqrt{3}}{2}\cdot y_5 = y_6 \tag{16}$$

$$\cos 3^\circ = 4\cos^3 1^\circ - 3\cos 1^\circ \tag{17}$$

$$4z^3 - 3z - x_6 = 0 \tag{18}$$

Now, let's use Cardano:

$$x^3 + px + q = 0 \tag{19}$$

$$p = -\frac{3}{4} \tag{20}$$

$$q = -\frac{1}{4}\cdot\left[\frac{\sqrt{3}}{2}\cdot\sqrt{\frac{a_4+1}{2}}\cdot\left(4\cdot\frac{a_4+1}{2}-3\right) + \frac{1}{2}\cdot\sqrt{\frac{1-a_4}{2}}\left(3-4\cdot\frac{1-a_4}{2}\right)\right] \tag{21}$$

$$q = -\frac{\sqrt{6}}{16}\cdot\sqrt{a_4+1}\cdot(2a_4-1) - \frac{\sqrt{2}}{16}\cdot\sqrt{1-a_4}\cdot(1+2a_4) \tag{22}$$

$$q=-\frac{\sqrt{6}}{16}\cdot\sqrt{\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{5}-1}{4}+1}{2}}+1}{2}}+1}\cdot\left(2\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{5}-1}{4}+1}{2}}+1}{2}}-1\right)- \tag{23}$$

$$-\frac{\sqrt{2}}{16}\cdot\sqrt{1-\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{5}-1}{4}+1}{2}}+1}{2}}}\cdot\left(1+2\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{5}-1}{4}+1}{2}}+1}{2}}\right) \tag{24}$$

$$\frac{q^2}{4}+\frac{p^3}{27}=\frac{q^2}{4}-\frac{1}{64}=\frac{16q^2-1}{64} \tag{25}$$

$$w=-\frac{q}{2}+i\cdot\frac{\sqrt{1-16q^2}}{8}=z^3\,;\ \arg w=\theta \tag{26}$$

$$W=\overline{w}=Z^3\,;\ \arg W=\overline{\theta} \tag{27}$$

$$|w|^2=\frac{q^2}{4}+\frac{1-16q^2}{64}=\frac{1}{64}\Rightarrow|z|=\frac{1}{2} \tag{28}$$

$$\tan\theta=-\frac{\sqrt{1-16q^2}}{4q}\,;\ \overline{\theta}=360^\circ-\theta \tag{29}$$

$$\arg z\in\{\varphi_1,\varphi_1+120^\circ,\varphi_1+240^\circ\}\,;\ \varphi_1=\frac{\theta}{3} \tag{30}$$

$$\sqrt[3]{w}=\frac{1}{2}\cdot\exp i(\varphi_1+k_1) \tag{31}$$

$$\arg Z\in\{\varphi_2,\varphi_2+120^\circ,\varphi_2+240^\circ\}\,;\ \varphi_2=\frac{360^\circ-\theta}{3} \tag{32}$$

$$=\{120^\circ-\varphi_1,240^\circ-\varphi_1,-\varphi_1\} \tag{33}$$

$$\sqrt[3]{W}=\frac{1}{2}\cdot\exp i(k_2-\varphi_1) \tag{34}$$

$$x_1=\frac{1}{2}\cdot\exp i(1^\circ)+\frac{1}{2}\cdot\exp i(-1^\circ)=\cos 1^\circ \tag{35}$$

$$x_2=\exp i(120^\circ)\cdot\frac{1}{2}\cdot\exp i(1^\circ)+\exp i(240^\circ)\cdot\frac{1}{2}\cdot\exp i(-1^\circ)=-\cos 59^\circ \tag{36}$$

$$x_3=\exp i(240^\circ)\cdot\frac{1}{2}\cdot\exp i(1^\circ)+\exp i(120^\circ)\cdot\frac{1}{2}\cdot\exp i(-1^\circ)=-\cos 61^\circ \tag{37}$$

$$\cos\frac{t}{3}=\frac{1}{2}\cdot\text{principal value of }\sqrt[3]{\cos 3t+i\cdot\sin 3t}+\frac{1}{2}\cdot\text{principal value of }\sqrt[3]{\cos 3t+i\cdot\sin 3t} \tag{38}$$

$$\sin\frac{t}{3}=\frac{1}{2}\cdot\text{principal value of }\sqrt[3]{-\sin 3t+i\cdot\cos 3t}+\frac{1}{2}\cdot\text{principal value of }\sqrt[3]{-\sin 3t-i\cdot\cos 3t}\ \blacksquare \tag{39}$$

2 Trisection Theorems

$$x^3-\frac{3}{4}\cdot x-\frac{\cos 3t}{4}=0\Leftrightarrow x\in\{\cos t,\cos(t+120^\circ),\cos(t+240^\circ)\} \tag{40}$$

$$x^3-\frac{3}{4}\cdot x+\frac{\sin 3t}{4}=0\Leftrightarrow x\in\{\cos(90^\circ-t),\cos(210^\circ-t),\cos(330^\circ-t)\} \tag{41}$$

$$x\in\{\sin t,\sin(t-120^\circ),\sin(t-240^\circ)\} \tag{42}$$

$$x^3+\frac{3\sin 3t}{4}\cdot x^2+\frac{12\sin^2 3t-27}{64}\cdot x+\frac{\sin^3 3t}{64}=0\Leftrightarrow x\in\{\sin^3 t,\sin^3(t-120^\circ),\sin^3(t-240^\circ)\} \tag{43}$$

$$x^3-\frac{3\cos 3t}{4}\cdot x^2+\frac{12\cos^2 3t-27}{64}\cdot x-\frac{\cos^3 3t}{64}=0\Leftrightarrow x\in\{\cos^3 t,\cos^3(t+120^\circ),\cos^3(t+240^\circ)\} \tag{44}$$

Proof of the last equation:

$$\blacktriangleright \ (a+bi)^3 = \cos 3t + i \sin 3t = k + si \tag{45}$$

$$a^3 - 3ab^2 = k \Rightarrow b^2 = \frac{a^3-k}{3a} \tag{46}$$

$$3a^2b - b^3 = s \tag{47}$$

$$b^2(3a^2-b^2)^2 = s^2 \tag{48}$$

$$\frac{a^3-k}{3a} \cdot \frac{(9a^3-a^3+k)^2}{9a^2} = s^2\,;\ a^3 = x \tag{49}$$

$$(x-k)(8x+k)^2 = 27s^2x \tag{50}$$

$$(x-k)(64x^2+16kx+k^2) = 27x - 27k^2x \ \blacksquare \tag{51}$$

Exercise 1. $(a+bi)^3 = \exp i(270° - 3t)$.

Exercise 2. $\cos^a(t/3)$ *and* $\sin^a(t/3)$.

Example 1. $x^3 + 3/8 \cdot x^2 - 3/8 \cdot x + 1/512 = 0$.

Example 2. $x^3 + 3/8 \cdot \sqrt{3} \cdot x^2 - 9/32 \cdot x + 3/512 \cdot \sqrt{3} = 0$.

3 Any Degree — n-Section Theorem

The equations below are soluble by radicals:

$$\cos\frac{t}{n} = \frac{1}{2} \cdot \text{principal value of } \sqrt[n]{\cos nt + i \cdot \sin nt} + \frac{1}{2} \cdot \text{principal value of } \sqrt[n]{\cos nt + i \cdot \sin nt} \tag{52}$$

$$\sin\frac{t}{n} = \frac{1}{2} \cdot \text{principal value of } \sqrt[n]{-\sin nt + i \cdot \cos nt} + \frac{1}{2} \cdot \text{principal value of } \sqrt[n]{-\sin nt - i \cdot \cos nt} \tag{53}$$

$$n > 0 \Rightarrow \cos nt = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \cdot n \cdot \frac{(n-i-1)!}{i!(n-2i)!} \cdot 2^{n-2i-1} \cdot (\cos t)^{n-2i} = p(\cos t) \tag{54}$$

$$p_n(x) = 0 \Leftrightarrow p(x) = \cos nt \Leftrightarrow x \in \left\{ \cos\left(t + \frac{2k\pi}{n}\right) ; 0 \le k \le n-1 \right\} \tag{55}$$

$$q_n(x) = 0 \Leftrightarrow p(x) = \cos\left(\frac{n\pi}{2} - nt\right) \Leftrightarrow x \in \left\{ \sin\left(t - \frac{2k\pi}{n}\right) ; 0 \le k \le n-1 \right\} \tag{56}$$

$$p_n(x) = \sum_{i=0}^{n} \alpha(n,i) \cdot x^i \tag{57}$$

Example 3. $\cos 2t = 2x^2 - 1\,;\ \cos 3t = 4x^3 - 3x\,;\ \cos 4t = 8x^4 - 8x^2 + 1\,;\ \cos 5t = 16x^5 - 20x^3 + 5x$.

Study with more details on this link to WikiPedia.

Cossines's and Chebyshev's triangle:

$$\begin{pmatrix} -\cos 0t + 1 \\ -\cos 1t & 1 \\ -\cos 2t - 1 & 0 & 2 \\ -\cos 3t & -3 & 0 & 4 \\ -\cos 4t + 1 & 0 & -8 & 0 & 8 \\ -\cos 5t & 5 & 0 & -20 & 0 & 16 \\ -\cos 6t - 1 & 0 & 18 & 0 & -48 & 0 & 32 \\ -\cos 7t & -7 & 0 & 56 & 0 & -112 & 0 & 64 \end{pmatrix} \tag{58}$$

Exercise 3. $\cos 1t = x\,;\ \cos 0t = 1$.

Exercise 4. $\cos^a(t/n)$.

Exercise 5. $\sin^a(t/n)$.

Sines's and Chebyshev's triangle:

$$\begin{pmatrix} -\cos 0t+1 \\ -\sin 1t & 1 \\ +\cos 2t-1 & 0 & 2 \\ +\sin 3t & -3 & 0 & 4 \\ -\cos 4t+1 & 0 & -8 & 0 & 8 \\ -\sin 5t & 5 & 0 & -20 & 0 & 16 \\ +\cos 6t-1 & 0 & 18 & 0 & -48 & 0 & 32 \\ +\sin 7t & -7 & 0 & 56 & 0 & -112 & 0 & 64 \end{pmatrix} \tag{59}$$

Exercise 6. *To construct two tetrahedra for* $\cos^a(t/n)$ *and* $\sin^a(t/n)$. *For each* (a,n), *there is a polynomial.*

4 Girard's Equations on Trisection

$$\cos t + \cos(t+120^\circ) + \cos(t+240^\circ) = 0 \tag{60}$$

$$\cos t\cos(t+120^\circ) + \cos t\cos(t+240^\circ) + \cos(t+120^\circ)\cos(t+240^\circ) = -\frac{3}{4} \tag{61}$$

$$\cos t\cos(t+120^\circ)\cos(t+240^\circ) = \frac{\cos 3t}{4} \tag{62}$$

$$\sin t + \sin(t-120^\circ) + \sin(t-240^\circ) = 0 \tag{63}$$

$$\sin t\sin(t-120^\circ) + \sin t\sin(t-240^\circ) + \sin(t-120^\circ)\sin(t-240^\circ) = -\frac{3}{4} \tag{64}$$

$$\sin t\sin(t-120^\circ)\sin(t-240^\circ) = -\frac{\sin 3t}{4} \tag{65}$$

He who trisects $3t$, also trisects $270^\circ - 3t$ and finds $\{t+30^\circ, t+150^\circ, t+270^\circ, t, t+120^\circ, t+240^\circ\}$ too.

5 Pentasection — Theorem and Girard

The equations below are soluble by radicals:

$$x^5 - \frac{20}{16}\cdot x^3 + \frac{5}{16}\cdot x - \frac{\cos 5t}{16} = 0 \Leftrightarrow x \in \{\cos t, \cos(t+72^\circ), \cos(t+144^\circ), \cos(t+216^\circ), \cos(t+288^\circ)\} \tag{66}$$

$$x^5 - \frac{20}{16}\cdot x^3 + \frac{5}{16}\cdot x - \frac{\sin 5t}{16} = 0 \Leftrightarrow x \in \{\sin t, \sin(t-72^\circ), \sin(t-144^\circ), \sin(t-216^\circ), \sin(t-288^\circ)\} \tag{67}$$

$$0 = S_1 = \sum c_i = S_1' = \sum s_i \tag{68}$$

$$-\frac{20}{16} = S_2 = \sum c_i c_j = S_2' = \sum s_i s_j \tag{69}$$

$$0 = S_3 = S_3' \tag{70}$$

$$\frac{5}{16} = S_4 = S_4' \tag{71}$$

$$-\frac{\cos 5t}{16} = \cos t\cos(t+72^\circ)\cos(t+144^\circ)\cos(t+216^\circ)\cos(t+288^\circ) \tag{72}$$

$$-\frac{\sin 5t}{16} = \sin t\sin(t-72^\circ)\sin(t-144^\circ)\sin(t-216^\circ)\sin(t-288^\circ) \tag{73}$$

Exercise 7. $\cos^a(t/5)$.

Sketch of solution:

$$x^5 = \frac{20}{16}\cdot x^3 - \frac{5}{16}\cdot x + \frac{\cos 5t}{16} \quad (74)$$

$$x - \cos^5 t = x - \frac{20}{16}\cdot \cos^3 t + \frac{5}{16}\cdot \cos t - \frac{\cos 5t}{16} \quad (75)$$

$$t' = t + 72^\circ \quad (76)$$

$$(x-\cos^5 t)(x-\cos^5 t')(x-\cos^5 t'')(x-\cos^5 t''')(x-\cos^5 t'''') = x^5 - \sigma_1 x^4 + \sigma_2 x^3 - \sigma_3 x^2 + \sigma_4 x - \sigma_5 = 0 \quad (77)$$

$$a+b+c+d+e = 0 \quad (78)$$

$$ab + \cdots + de = -20/16 \quad (79)$$

$$abc + \cdots + cde = 0 \quad (80)$$

$$abcd + \cdots + bcde = 5/16 \quad (81)$$

$$abcde = -1/16 \cdot \cos 5t \quad (82)$$

$$a^5 + b^5 + c^5 + d^5 + e^5 = \sigma_1 \quad (83)$$

$$a^5 b^5 + \cdots + d^5 e^5 = \sigma_2 \quad (84)$$

$$a^5 b^5 c^5 + \cdots + c^5 d^5 e^5 = \sigma_3 \quad (85)$$

$$a^5 b^5 c^5 d^5 + \cdots + b^5 c^5 d^5 e^5 = \sigma_4 \quad (86)$$

$$(abcde)^5 = \sigma_5 = -1/2^{20} \cdot \cos^5 5t \quad (87)$$

$$(88)$$

Exercise 8. $\sin^a(t/5)$.

He who pentasects $5t$, also pentasects $450^\circ - 5t$ and finds $t + k^\circ\,;\ k \in \{0, 54, 72, 126, 144, 198, 216, 270, 288, 342\}$ too.

6 $(x - r_1^a)\cdots(x - r_n^a) = 0$

Theorem 1. *Let the polynomial equation be $p(x) = 0\,;\ \deg p = n$, whose roots are r_i. Let $a \geq 2$ be a natural. Then, it's always possible to construct another polynomial $q(x) = 0$, with the same degree n, whose roots are r_i^a, without solving the equation $p(x) = 0$ and only by using Girard's relations.*

7 Heptasection

The equations below are soluble by radicals:

$$x^7 - \frac{112}{64}\cdot x^5 + \frac{56}{64}\cdot x^3 - \frac{7}{64}\cdot x - \frac{\cos 7t}{64} = 0 \Leftrightarrow x \in \left\{\cos\left(t + \frac{2k\pi}{7}\right)\ ;\ 0 \leq k \leq 6\right\} \quad (89)$$

$$x^7 - \frac{112}{64}\cdot x^5 + \frac{56}{64}\cdot x^3 - \frac{7}{64}\cdot x + \frac{\sin 7t}{64} = 0 \Leftrightarrow x \in \left\{\sin\left(t - \frac{2k\pi}{7}\right)\ ;\ 0 \leq k \leq 6\right\} \quad (90)$$

Main Corollary. $A = \{\cos^a(t/q)\,;\ \sin^a(t/q)\,|\, t \in \arccos\mathbb{Q}\,;\ q \in \mathbb{Q}\,;\ a \in \mathbb{N}\}$ *is a subset of the algebrical numbers.*

Exercise 9. *Girard:*

$$0 = S_1 \quad (91)$$

$$-\frac{112}{64} = S_2 \quad (92)$$

$$0 = S_3 \quad (93)$$

$$\frac{56}{64} = S_4 \quad (94)$$

$$0 = S_5 \quad (95)$$

$$-\frac{7}{64} = S_6 \quad (96)$$

$$\frac{\cos 7t}{64} = P \quad (97)$$

Exercise 10. $\cos^a(t/7)$.

Exercise 11. $\sin^a(t/7)$.

He who heptasects $7t$, also heptasects $630°-7t$ and finds $t+k°$; $k \in \{0, 90/7, 360/7, 450/7, 720/7, 810/7, 1080/7, 1170/7, 1440/7, 1530/7, 1800/7, 270, 2160/7, 2250/7\}$ too.

8 Girard's Equations on n-section

$$S_1 = \sum c_i = -\frac{\alpha(n, n-1)}{2^{n-1}} \quad (98)$$

$$S_2 = \sum c_i c_j = +\frac{\alpha(n, n-2)}{2^{n-1}} \quad (99)$$

$$S_i = (-1)^i \cdot \frac{\alpha(n, n-i)}{2^{n-1}} \quad (100)$$

$$S_n = P = (-1)^n \cdot \frac{\alpha(n, 0)}{2^{n-1}} \quad (101)$$

Exercise 12. *To construct a polygon of* $n = 11$ *sides.* $n = 13, 17, 19, 23, \cdots$

$$\cos nt = +\, 2^{n-1} \cdot x^n + 0 \quad (102)$$

$$-\, n \cdot 2^{n-3} \cdot x^{n-2} + 0 \quad (103)$$

$$+\, n(n-3)/2 \cdot 2^{n-5} \cdot x^{n-4} + 0 \quad (104)$$

$$-\, n(n-4)(n-5)/6 \cdot 2^{n-7} \cdot x^{n-6} + 0 \quad (105)$$

$$+\, n(n-5)(n-6)(n-7)/24 \cdot 2^{n-9} \cdot x^{n-8} + 0 \quad (106)$$

$$-\, n(n-6)(n-7)(n-8)(n-9)/5! \cdot 2^{n-11} \cdot x^{n-10} + 0 \quad (107)$$

$$+\, n(n-7, 8, 9, 10, 11)/6! \cdot 2^{n-13} \cdot x^{n-12} + 0 \quad (108)$$

$$-\, n(n-8, 9, 10, 11, 12, 13)/7! \cdot 2^{n-15} \cdot x^{n-14} + 0 \quad (109)$$

$$+\, n(n-9, 10, 11, 12, 13, 14, 15)/8! \cdot 2^{n-17} \cdot x^{n-16} + 0 \quad (110)$$

$$-\, n(n-10, 11, 12, 13, 14, 15, 16, 17)/9! \cdot 2^{n-19} \cdot x^{n-18} + 0 \quad (111)$$

$$+\, n(n-11, 12, 13, 14, 15, 16, 17, 18, 19)/10! \cdot 2^{n-21} \cdot x^{n-20} + 0 \quad (112)$$

$$-\, n(n-12, 13, 14, 15, 16, 17, 18, 19, 20, 21)/11! \cdot 2^{n-23} \cdot x^{n-22} + 0 \quad (113)$$

9 $\cos z \in \mathbb{C}$

$$\cos z = a + bi \tag{114}$$
$$\cos(x + yi) = \cos x \cosh y - i \sin x \sinh y \tag{115}$$
$$\cos x \cosh y = a \tag{116}$$
$$-\sin x \sinh y = b \tag{117}$$
$$\cosh^2 y - \sinh^2 y = 1 = \frac{a^2}{\cos^2 x} - \frac{b^2}{\sin^2 x}; \; u = \cos^2 x \tag{118}$$
$$1 = \frac{a^2}{u} + \frac{b^2}{u-1} \tag{119}$$
$$u^2 - u = a^2(u-1) + b^2 u \tag{120}$$
$$u^2 + u(-a^2 - b^2 - 1) + a^2 = 0 \tag{121}$$
$$\Delta = (a^2 + b^2 + 1)^2 - 4a^2 \tag{122}$$
$$\cos^2 x = \frac{a^2 + b^2 + 1 \pm \sqrt{\Delta}}{2} \tag{123}$$
$$\cos x \in \{c_1, c_2, c_3, c_4\} \tag{124}$$
$$x \in \arccos c_i + 2\pi \cdot \mathbb{Z} \tag{125}$$
$$\cosh y = \frac{a}{c_i} \tag{126}$$

Example 4. $\arccos(3 + 4i) = w$.

$$\cos^2 x = 13 \pm \sqrt{13^2 - 9} \tag{127}$$
$$\cos x \in \{c_1, c_2, c_3, c_4\} \tag{128}$$
$$\cosh y = \frac{3}{c} \tag{129}$$
$$x + yi = \arccos \sqrt{13 + 4\sqrt{10}} + i \cdot \operatorname{argcosh} \left(\frac{3}{\sqrt{13 + 4\sqrt{10}}} \right) \tag{130}$$

Exercise 13. $\cos nz = 3 + 4i = \sin nw$.

10 Third Degree — Reduction of order

$$x^3 - Sx^2 + Qx - P = 0 \Leftrightarrow \begin{cases} x + y + z = S \\ x^2 + y^2 + z^2 = S^2 - 2Q \\ x^3 + y^3 + z^3 = S^3 - 3SQ + 3P \end{cases}$$

$$z = S - x - y \tag{131}$$
$$w = x + y \tag{132}$$
$$x^2 + y^2 + \cancel{S^2} - 2Sw + w^2 = \cancel{S^2} - 2Q \tag{133}$$
$$x^2 + y^2 - 2Sx - 2Sy + x^2 + 2xy + y^2 = -2Q \tag{134}$$

$$2y^2 + y(-2S + 2x) + 2x^2 - 2Sx + 2Q = 0 \tag{135}$$
$$y^2 + y(x - S) + (x^2 - Sx + Q) = 0 \tag{136}$$
$$\Delta = x^2 - 2Sx + S^2 - 4x^2 + 4Sx - 4Q \tag{137}$$

Theorem 2. *If we know a single root of the equation $x^3 - Sx^2 + Qx - P = 0$, then the other two roots are:*

$$f_1^2(x) = \frac{S - x \pm \sqrt{\Delta}}{2}, \tag{138}$$

where $\Delta = -3x^2 + 2Sx + S^2 - 4Q$.

Example 5. *We know* $\cos 1°$.

$$x^3 + px + q = 0 \tag{139}$$

$$S = 0\,;\, Q = -\frac{3}{4}\,;\, P = -\cos 3° \tag{140}$$

$$\cos^2 1° + \cos^2 59° + \cos^2 61° = \frac{3}{2} \tag{141}$$

$$\cos^3 1° - \cos^3 59° - \cos^3 61° = \frac{3}{4} \cdot \cos 3° \tag{142}$$

$$x_0 = \cos 1° \tag{143}$$

$$\Delta = -3\cos^2 1° + 3 \tag{144}$$

$$y = \frac{-x \pm \sqrt{\Delta}}{2} \tag{145}$$

$$-\cos 61° = \frac{-\cos 1° + \sqrt{3 - 3\cos^2 1°}}{2} \tag{146}$$

$$-\cos 59° = \frac{-\cos 1° - \sqrt{3 - 3\cos^2 1°}}{2} \tag{147}$$

11 Any Degree — Reduction of Order

Theorem 3. *By Galois's theorem, let $a \leq 4$. If we know $b = n - a$ roots of the equation*

$$x^n - S_1 x^{n-1} + \cdots + (-1)^{n-1} S_{n-1} x + (-1)^n S_n = 0, \tag{148}$$

then the other a roots are:

$$f_{n-a}^i(x_1, \cdots, x_{n-a}) \in \{y_1, \cdots, y_a\}, \forall i \in \{1, \cdots, a\}. \tag{149}$$

Proof: ▶ Divide the original $p = (1, -S_1, +S_2, \cdots)$ by $(x - x_1)$ to obtain $q = (1, -S_1', +S_2', \cdots)$, where $S_1' = S_1 - x_1$, $S_2' = S_2 - x_1 S_1'$ and so on. Now, solve $q(x) = 0$. ■

12 Variation of Parameters — From Polynomials to Equivalent Exponentials

$$\exp r_i t = 1 f_1(t) + r_i f_2(t) + \frac{r_i^2}{2} \cdot f_3(t) + \cdots + \frac{r_i^n}{n!} \cdot f_{n+1}(t) \tag{150}$$

$$E = T \cdot F \tag{151}$$

$$\left[E^\top\right]_{1\times n} = [F^\top]_{1\times n} \cdot \left[T^\top\right]_{n\times n} \tag{152}$$

$$[E']^\top = [F']^\top \cdot T^\top \text{ and all other } n \text{ derivatives} \tag{153}$$

$$\left[E^\top\right]_{n\times n} = [\exp M]_{n\times n} \cdot \left[T^\top\right]_{n\times n} \tag{154}$$

We have polynomial of n-th degree of x with real coefficients equals to zero.

$$p_n(x) = 0 \Leftrightarrow x \in \{r_1, \cdots, r_n\} = R \tag{155}$$

We have exponential of xt plus polynomial of $(n-1)$-th degree of x with coefficients $f_i(t)$ equals to zero.

$$\varphi_{n-1}(x,t) = e^{xt} + p'_{n-1}(x,t) \tag{156}$$
$$\varphi_{n-1}(x,t) = 0 \Leftarrow x \in R, \forall t \in \mathbb{R} \tag{157}$$
$$\varphi_{n-1}^{-1}(0) \supset R \times \mathbb{R} \tag{158}$$

And for each term a of the Taylor series $\exp rt = \sum \frac{r^a}{a!} \cdot t^a$,

$$\psi_a(x,a) = x^a + p''_{n-1}(x,n) \tag{159}$$
$$\psi_a(x,a) = 0 \Leftarrow x \in R, \forall a \in \mathbb{N} \tag{160}$$
$$\psi_a^{-1}(0) \supset R \times \mathbb{N} \tag{161}$$

13 Hungerford's Algebra

Definition 1. *Let E and F be extension fields of a field K. A nonzero map $\sigma : E \to F$ which is both a field and a K-module homomorphism is called a K-homomorphism.*

Definition 2. *If a field automorphism $\sigma \in Aut\ F$ is a K-homomorphism, then σ is called a K-automorphism of F.*

Definition 3. *The group of $\{\sigma\,;\ \sigma$ is K-automorphism of $F\}$ is called the Galois group of F over K and is denoted $Aut_K\,F$.*

Definition 4. *Let F be a field and $f \in F[x]$ a polynomial of positive degree. f is said to split in $F[x]$ if f can be written as a product of linear factors in $F[x]$; that is, $f = u_0(x-u_1)\cdots(x-u_n)$, with $u_i \in F$.*

Definition 5. *Let K be a field and $f \in K[x]$ a polynomial of positive degree. An extension field F of K is said to be an splitting field over K of the polynomial f if f splits in $F[x]$ and $F = K(u_1, \cdots, u_n)$, where u_i are the roots of f in F.*

Definition 6. *Let K be a field. The Galois group of a polynomial $f \in K[x]$ is the group $Aut_K\,F$, where F is a splitting field of f over K.*

Theorem 4. *The Galois group of $f(x) = x^p - x - a = 0$ is S_p, the group of permutations of $\{1, 2, \cdots, p\}$.*

Theorem 5. *The Galois group of $x^5 - 4x + 2 = 0$ is S_5.*

Definition 7. $A_n = \{\sigma \in S_n; \sigma$ *is even*$\}$

Definition 8. *The commutator of A_n is defined as $A'_n = \{aba^{-1}b^{-1}\,;\ a, b \in A_n\}$.*

Theorem 6. *A_5 is not solvable.*

Proof: ▶ $ab = ba \Rightarrow aba^{-1}b^{-1} = e$
A_5 is not abelian. $A'_5 \lhd A_5$. A_5 is simple.
Therefore, $A'_5 = A_5$. If $A_5^{(n)}$ were eventually equal to (e), then A_5 would be solvable.
Therefore, A_5 is not solvable. ■

Theorem 7. *$A_5 < S_5$ is not solvable.*

Theorem 8. *Let K be a field and $f \in K[x]$ a polynomial of degree $n > 0$, where char K does not divide $n!$ (which is always true when char $K = 0$). Then the equation $f(x) = 0$ is solvable by radicals if and only if the Galois group of f is solvable.*

quod erat demonstrandum — Out of charity, there is no salvation at all.

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